

DERIVING THE ALGEBRAIC RECONSTRUCTION TECHNIQUE (ART) BY THE METHOD OF PROJECTIONS ONTO CONVEX SETS (POCS)

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1. INTRODUCTION

Very large systems of linear equations can only be solved by iterative methods. The most popular of which (the Gauss-Seidel and Jacobi methods) do not always converge. By considering each equation of a linear system as a convex set we have used the method of POCS [1] to derive an iterative method that always converges. It turned out that this method is identical to ART (also known as the Kaczmarz method) which was derived by a different approach [2][3] and is largely used in computerized tomography (CT). Having shown that ART can be derived by the method of POCS, additional convex constraints can be used as well in CT such as the constraints of positivity and of compact support (i.e., zero value outside of a certain region). In the case of underdetermined systems, the more constraints are introduced, the better will be the approximation. The solution will also converge to the point belonging to the intersection of all convex sets which is the closest to the starting solution. The fact that ART is a particular case of POCS is, as far as we know, not exploited by CT community where additional constraints are considered to introduce a bias in ART and little attention is given to the initial solution.

2. DERIVING ART BY POCS

Considering the 2D case in the x_1 and x_2 plane, we have two linear equations $a_{11}x_1 + a_{12}x_2 = y_1$ and $a_{21}x_1 + a_{22}x_2 = y_2$. From point (x_1^k, x_2^k) we want to find its closest point x_1^{k+1} and x_2^{k+1} on the line $a_{11}x_1 + a_{12}x_2 = y_1$. That is we want to minimize the distance:

$$f = \sqrt{(x_1^{k+1} - x_1^k)^2 + (x_2^{k+1} - x_2^k)^2} \quad (1)$$

subject to the constraint

$$g = a_{11}x_1^{k+1} + a_{12}x_2^{k+1} - y_1 = 0 \quad (2)$$

By the method of Lagrange multipliers we have:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad (3)$$

and by eliminating the multiplier λ :

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \quad (4)$$

which applied to (1) and (2) gives:

$$x_1^{k+1} = x_1^k - \frac{a_{11}(a_{11}x_1^k + a_{12}x_2^k - y_1)}{a_{11}^2 + a_{12}^2} \quad (5)$$

$$x_2^{k+1} = x_2^k - \frac{a_{12}(a_{11}x_1^k + a_{12}x_2^k - y_1)}{a_{11}^2 + a_{12}^2} \quad (6)$$

This point becomes the new point (x_1^k, x_2^k) and we now find its closest point on the second equation by the same method:

$$x_1^{k+1} = x_1^k - \frac{a_{21}(a_{21}x_1^k + a_{22}x_2^k - y_2)}{a_{21}^2 + a_{22}^2} \quad (7)$$

$$x_2^{k+1} = x_2^k - \frac{a_{22}(a_{21}x_1^k + a_{22}x_2^k - y_2)}{a_{21}^2 + a_{22}^2} \quad (8)$$

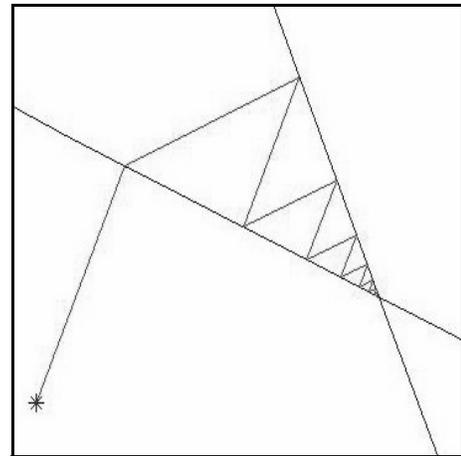


Fig. 1. Convergence of ART.

As illustrated in Fig. 1 this mechanism of alternate orthogonal projections is repeated until convergence to the solution. In this figure * indicates the initial solution

and the two lines are the linear equations whose solution is their intersection. An iteration is completed after projecting onto each convex sets (after having computed equations 5 to 8, in this example). For j unknowns and i equations the general equation becomes:

$$x_j^{k+1} = x_j^k \text{D} \frac{a_{ij}(\sum_j a_{ij}x_j^k \text{D} y_i)}{\sum_j a_{ij}^2} \quad (9)$$

which is ART [2][3].

The MATLAB code for solving eq. (9) for any system of the form $\mathbf{Ax} = \mathbf{y}$ or:

$$\sum_j a_{ij}x_j = y_i \quad (10)$$

for each i by eq. (9) is given here:

```
function x = pocsLs(A, y, niter);
    a2 = (sum((A.^2)'))'; % ' transpose
    dim = size(A);
    nx = dim(2); % n of unknowns
    ne = dim(1); % n of equations
    x = ones(nx, 1); % Initial solution
    for i = 1:niter
        for j = 1:ne
            C = (A(j,:) * x - y(j))/a2(j);
            x = x - A(j,:) * C;
        end
        % x = (x > 0) .* x; Positivity constraint.
        % Other convex constraints.
    end
```

If the system is determined, the additional constraints can speed the convergence. If not they are necessary to limit the solution space. 11 convex constraints can be found in [1]. Others can be derived by the method of Lagrange multipliers after having proven that they each form a convex set [1][4]. The positivity and compact support constraints are realized by setting to zero the negative values of x and the values of x outside the support [1].

3. NUMERICAL RESULTS

Algorithm (9) has been tested on the 128*128 pixel Shepp-Logan phantom (Fig. 2). The attenuation coefficients were computed for each point in each orientation by an exponential function taking into account the distance between a point and the limit of the phantom in the direction of the gamma camera (Fig. 3).



Fig. 2. The Shepp-Logan phantom

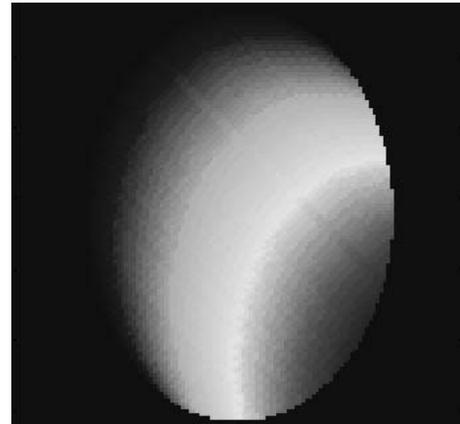


Fig. 3. Attenuation map at 45 degree.

The Radon transforms (185 pixels each) of this phantom were computed in 32 regular steps from 0 to 180 degrees. \mathbf{A} has thus 185x32 lines and 128x128 columns, \mathbf{x} has 128*128 lines and has \mathbf{y} 185*32 lines. This system is too large to be solved by the program PocsLs. Eq. (9) was thus evaluated as follows starting from the first orientation:

- The attenuation map for orientation θ is computed and multiplied point by point by the phantom.
- Three Radon transforms are computed for orientation θ : R1 for the image computed in the preceding step, R2 for the phantom and R3 for the map to the power of two.
- The values of $(R1 - R2)/R3$ are backprojected for orientation θ .
- The result is multiplied point by point by the attenuation map in θ and each pixel is updated by sub-

tracting this from its value at the preceding iteration.

- The positivity constraint is applied. (The compact support constraint is applied by the preceding step).
- Increase θ and restart from the first step.

These six steps constitute an iteration. Reconstruction with 10 iterations is shown in Fig. 4.

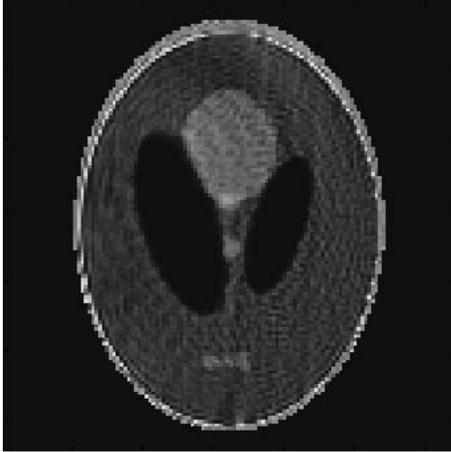


Fig. 4. Reconstruction of the phantom by ART

The image percent error is computed at each iteration by:

$$err_k = 100 \frac{\|Original - Reconstructed\|}{\|Original\|}$$

where $\|x\|$ is the euclidian norm of x , and is plotted in Fig. 5.

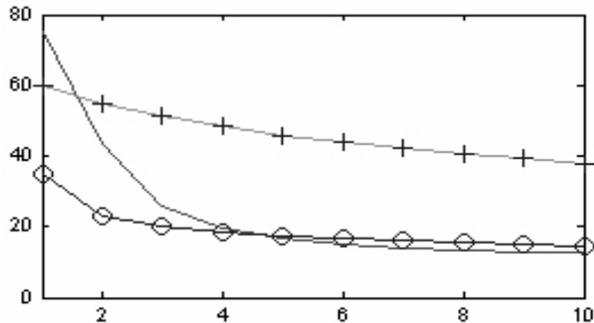


Fig. 5. Image percent error vs. iterations. Continuous line: ART; +: MLEM; o: OSEM

At iteration 10 the error is 12.72%. This is a quite good estimate of the solution considering that we have a system of 16,364 unknowns (128x128) with only 5,920 (185*32) equations. That is with only 35.13% of

the necessary number of equations to obtain the exact solution.

In Fig. 4 ART is also compared to the reconstructions obtained by MLEM [5] and its accelerated version OSEM (ordered subsets expectation maximization) with eight subsets [6]. As generally reported one iteration of OSEM is approximately equivalent to ten iterations of MLEM. The Fourier backprojections (FBP) method gives an error of 39.84% with an uniform attenuation.

Real data were also obtained with a clinical SPECT system equipped with an attenuation correction option. Reconstructions from the Jaszczack heart phantom and from patients data were obtained with ART and OSEM using ten iterations. Generally after 3 iterations the reconstructed images by both methods were barely discernible from each other by eye inspection. Our reconstructions were also very similar to those of the commercial system which are based on the OSEM approach like most other systems in the industry.

5. COMPARISON OF ART WITH MLEM-OSEM

The MLEM algorithm consists in solving:

$$x_j^{k+1} = x_j^k \sum_i \frac{a_{ij} y_i}{\sum_j a_{ij} x_j} \quad (11)$$

where the a_{ij} are normalized for each j :

$$\sum_i a_{ij} = 1 \quad (12)$$

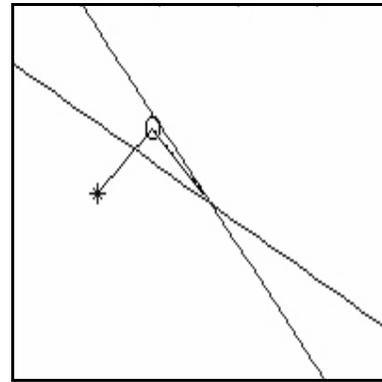


Fig. 6. Convergence of MLEM with 100 iterations. The point in O is reached at the first iteration.

In [7] MLEM (eq. (11)) was derived by POCS by minimizing the Kullback-Leibler (KL) pseudo-distance between y_i and $\sum_j a_{ij} x_j$ while minimizing the same distance between $\sum_j a_{ij} x_j$ and y_i gives the MART (multi-

plicative ART) algorithm. The convergence of MLEN for a 2x2 system is shown in Fig. 6. For large underdetermined systems if x^∞ is the closest to x^0 is still an open question [7].

6. CONCLUSION

The ART algorithm can be derived easily and rapidly by the method of POCS. Using more conventional mathematical methods it takes many pages to develop ART [2][3]. Since ART is POCS, additional convex constraints can be added. The initial solution can also influence the reconstruction because ART converges to the solution respecting eq. (10) which is the closest to it. Starting from an uniform image, for example, will maximize the entropy.

The POCS method is also interesting as a uniform framework to develop and compare CT algorithms at a more abstract level. Knowing that ART, MLEM and MART can all be derived by POCS by minimizing the euclidian distance, the KL distance between y_i and $\sum_j a_{ij}x_j$, and the KL distance between $\sum_j a_{ij}x_j$ and y_i respectively is of theoretical and practical interests.

More generally, Eq. (9), because of its guaranteed convergence, should be considered as the method for solving linear system *in lieu of* the Gauss-Seidel or Jacobi methods presented in almost all textbooks on numerical or applied mathematics. It can also be used with constraints and proper starting solution to solve any underdetermined linear systems of equations

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