Level Set Technique for Image Reconstruction

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Abstract: Level Set (LS) technique is shown to be an effective regularization technique in non-linear inverse problems because of its topological based representation of unknown structures using level zero (a front) of a higher dimension function (level set function). The representation of structures using the level set function is of great use to address the need of image reconstruction from ill-posed inverse problem containing limited set of available data and prior information. The cost functional of the classical LSRM is based on the quadratic formulations (least squares fitting or L2 norms) of data mismatch and regularization terms. However, the L2 norm optimization problems are not robust to outliers and measurement noise. To achieve a high robustness against outliers and noise, a general inverse problem can be formulated in terms of the L1 norms or a combination of the L1 norms and the L2 norms, instead of merely usage of the L2 norms. In this paper, we derive a novel LS based regularization method (LSRM) which allows any possible combinations of the L1 and the L2 norms on the inverse problem terms. To show the implementation of the derived LSRM, we use an ill-posed inverse problem called Electrical Impedance Tomography (EIT). The image reconstruction results of the proposed LSRM are compared to those of four state of the art regularization methods: Gauss-Newton (GN) with Tikhonov regularization term, GN with NOSER algorithm, Total Variation (TV), and the PDIPM. According to our results, the proposed LSRM produces more robust results in the presence of high level of noise (additive 60dB Gaussian noise) and strong outliers (loss of measurement data) when compared with the competing methods.

1 Introduction

Level set (LS) technique has gained a noticeable attention in a variety of different applications ranged from computer vision, image enhancement and segmentation to microchip fabrication. The application of LS in inverse problem was first proposed in 1996. Since then, LS was applied for image reconstruction in linear inverse problems, such as x-ray CT, PET, SPECT as well as in non-linear inverse problems, such as microwave imaging, non-destructive imaging, near-infrared imaging, and electrical impedance tomography (EIT). The representation of LS as part of a solution scheme for ill-posed inverse problems is desirable because of the high potential of LS in reconstructing 2D or 3D images from few available data [5]. The LS based regularization method (LSRM) considers the topological information of structures unknown, without the need of knowing the number and the origin of the structures, and reconstruct the structures using the evolution of a level set function which minimizes a predefined cost functional. Most recently in [7], we show the first clinical results of applying the LSRM in the image reconstruction of lungs using EIT system, which is an ill-posed inverse problem. In [7], we propose a LSRM using L2 norms on the inverse problem terms. Borsic and Adler (2012) show the L2 norms are sensitive to spatial noise and data loss (outliers). They propose the L1 norms as a more solid alternative and discuss the minimization of the L1 norm based inverse problem using primal-dual interior point method (PDIPM). They show the L1 norms provide higher robustness against outliers and noise when compared with L2 norm based regularization method, such as Gauss-Newton (GN) method. In this paper, we derive a novel framework to solve a L1 norm based inverse problem using the LSRM. The developed LSRM, hereinafter called LS-PDIPM, minimizes the L1 norms using the PDIPM. The proposed LS-PDIPM is highly successful in dealing with high level of spatial noise (60 dB Gaussian noise) and strong outliers. The performance of the proposed LS-PDIPM is compared with that of four state of the art regularization methods, Gauss-Newton (GN) with Tikhonov regularization term [8], GN with NOSER algorithm [4], Total Variation (TV)[3], and the PDIPM [2], over the same simulated data of EIT. Our results show the proposed LS-PDIPM is more robust against additive 60 dB Gaussian noise and strong data outliers in the comparison with the comparing methods.

2 Generalized PDIPM

A general primal problem can be written as follows

$$(P) = \text{argmin}\{\sum_{i=1}^{D_1} |f_{d_i}(m)| + \sum_{j=1}^{D_2} |f_{p_j}(m)| + \|g_d(m)\|^2 + \|g_p(m)\|^2\} \quad (1)$$

where $f_d(m)$ is a L1 norm based data mismatch term, $f_p(m)$ is a L1 norm based regularization term, $g_d(m)$ is a L2 norm

based data mismatch term, and $g_p(m)$ is a L2 norm based regularization term. A primal minimization problem can be formed through any combination of the error terms defined in (1). The non-differentiability of L1 norm is resolved through using a centering condition which is the replacement of the L1 norm by a quadratic norm using an auxiliary variable, called β. The Primal-Dual (PD) framework can be formulated as

$$C_d(m) = f_{d_i}(m) - x_{d_i}\sqrt{f_{d_i}(m)^2 + \beta} = 0,\ \forall i \quad (2)$$

$$C_p(m) = f_{p_j}(m) - x_{p_j}\sqrt{f_{p_j}(m)^2 + \beta} = 0,\ \forall j \quad (3)$$

$$|x_{d_i}| \leq 1, |x_{p_j}| \leq 1 \quad (4)$$

$$F_c(m) = \frac{\partial}{\partial m}(f_d(m))x_d + \frac{\partial}{\partial m}(f_p(m))x_p + \frac{\partial}{\partial m}(\|g_d(m)\|^2) + \frac{\partial}{\partial m}(\|g_p(m)\|^2) = 0 \quad (5)$$

The derived general Primal-Dual (PD) framework above is solved iteratively using an iterative method, such as Newton method as follows

$$\begin{bmatrix} \frac{\partial}{\delta m}F_c(m) & \frac{\partial}{\delta x_d}F_c(m) & \frac{\partial}{\delta x_p}F_c(m) \\ \frac{\partial}{\delta m}C_d(m) & \frac{\partial}{\delta x_d}C_d(m) & \frac{\partial}{\delta x_p}C_d(m) \\ \frac{\partial}{\delta m}C_p(m) & \frac{\partial}{\delta x_d}C_p(m) & \frac{\partial}{\delta x_p}C_p(m) \end{bmatrix} \begin{bmatrix} \delta m \\ \delta x_d \\ \delta x_p \end{bmatrix} =$$

$$- \begin{bmatrix} F_c(m) \\ f_d(m) - (\sqrt{f_d(m)^2 + \beta})x_d \\ f_p(m) - (\sqrt{f_p(m)^2 + \beta})x_p \end{bmatrix} \quad (6)$$

where two auxiliary variables x_d and x_p are in the range $[-1, 1]$ depending on the absolute value of $f_d(m)$ and $f_p(m)$; respectively. The primal variables (m) are updated with a line search procedure, as $m^{(k+1)} = m^{(k)} + \lambda_m \delta m^{(k)}$, where k is the iteration number, δm is the update value with a descend direction to the optimal point, and λ is the step length [6]. In a similar manner, the dual variables (x_d and x_p) are also updated using *scaling rule* as follows $x^{(k+1)} = x^{(k)} + \min(1, \varphi^*)\, \delta x^{(k)}$[1], where φ^* is a scalar value such that $\varphi^* = \sup\left\{\varphi : \left|x_i^{(k)} + \varphi\, \delta x_i^{(k)}\right| \leq 1, \quad i = 1, \ldots, n\right\}$.

3 Level Set

In this section, we formulate the solution of a system defined as (1) using the proposed level set based PD-IPM, hereinafter called LS-PDIPM. The solution of the L1L2 problem is derived using the LS-PDIPM. The solutions for the L2L1 and L1L1 problems are analogous. In [2], Borsic and Adler detail the PDIPM framework when applied for EIT.

3.1 LS-PDIPM for the L1L2 Problem

The level set function (Ψ) is a signed distance function which is zero at the optimal solution and nonzero otherwise. The minimum distance from the optimal solution is achieved at zero level set function. The evolution of the level set function according to the minimization of a functional objective function (primal problem), which can be a standard least square error function, results in the optimal solution of an inverse problem. A mapping function (Φ) is used to project the level set function onto finite element mesh (FEM). The level set evolution function is as follows $\Psi_{k+1} = \Psi_k + \lambda(\Delta\Psi)$, where Ψ_{k+1} is the updated level set function, Ψ_k is the current level set function, $\Delta\Psi$ is the update, Φ is the mapping function, λ is the step size.

The primal formulation (P) for the L1L2 problem is a special state of the general primal problem in (1) when $|f_p(m)| = \|g_d(m)\|^2 = 0$ and can be written as

$$(P) = \underset{\Phi(\Psi)}{\operatorname{argmin}}\{\sum_i W_i|h_i(\Phi(\Psi)) - d_i| + \alpha\|L(\Phi(\Psi) - \Phi(\Psi^0))\|^2\} \quad (7)$$

where $|f_d(m)$ is replaced by $f_d(\Phi(\Psi)) = W|h(\Phi(\Psi)) - d|$, $|g_p(m)$ is replaced by $g_p(\Phi(\Psi)) = \alpha\|L(\Phi(\Psi) - \Phi(\Psi^0))\|^2$, W is a weighting diagonal matrix, W_i is the i-th diagonal element, $h_i(\Phi(\Psi))$ is the i-th forward measurement, d_i is the i-th measured data, L is the regularization matrix, $\Phi(\Psi)$ is the model parameter distribution or the primal variables, $\Phi(\Psi^0)$ is a reference model parameter distribution According to the chain rule, the LS Jacobian matrix (J_{LS}) can be written as below $J_{LS} = \frac{\partial d}{\partial \Psi} = (\frac{\partial h}{\partial \Phi(\Psi)})(\frac{\partial \Phi(\Psi)}{\partial \Psi}) = (J_{GN})(M)$. To make the algorithm computationally efficient, we restrict the Jacobian computation within a narrow band containing the data (non-zeros). To construct the narrow band, we define the level set function, or the signed distance function, to be negative inside its boundary and positive outside. Matrix M is non-zero within the narrow band and zero otherwise, which is the notion of Dirac delta function. In every iteration of the level set function (Ψ), we calculate the Jacobian matrix (J_{LS}) for the narrow band. We define a dual variable x_i in the range $[-1, 1]$, depending on the absolute value of $W_i(d - h(\Phi(\Psi)))$. The dual problem can be written as

$$(D) = \underset{\Phi(\Psi)}{\operatorname{argmin}}\{max_x \quad x^T W(h(\Phi(\Psi)) - d) + \alpha\|L(\Phi(\Psi) - \Phi(\Psi^0))\|^2\}, with|x| \leq 1 \quad (8)$$

we define $\left[x^T W(h(\Phi(\Psi)) - d)\right] = D_1$. Taking the first order derivative of D_1 with respect to the level set function results in

$$\frac{\partial}{\partial \Psi}[D_1] = \frac{\partial}{\partial \Phi(\Psi)}[D_1]\frac{\partial}{\partial \Psi}[\Phi(\Psi)] = J_{GN}^T M^T W x, \quad (9)$$

where $J_{GN}^T M^T W x = J_{LS}^T W x$. We define $\|L(\Phi(\Psi) - \Phi(\Psi^0))\|^2 = D_2$. Taking the first order derivative of D_2 with respect to the level set function gives

$$\frac{\partial}{\partial \Psi}[D_2] = 2L^T L(\Phi(\Psi) - \Phi(\Psi^0))M = 2\,\mathrm{M}^T L^T L(\Phi(\Psi) - \Phi(\Psi^0)) \quad (10)$$

so the first order condition for the minimization in the dual problem is $J_{LS}^T(\Phi(\Psi))Wx + 2\alpha M^T L^T L(\Phi(\Psi) - \Phi(\Psi^0)) = 0$. Nulling the difference between the primal and dual problems gives us the following Primal-Dual gap

$$G_{PD} = \sum_{i=1}^{D}\{|W_i(h(\Phi(\Psi))_i - d_i)| - x_i W_i(h(\Phi(\Psi)) - d)\} \quad (11)$$

the primal-dual G_{PD} is null if, for each i, either $W_i(h(\Phi(\Psi)) - d) = 0$ or $x_i = W_i(h(\Phi(\Psi)) - d)/|W_i(h(\Phi(\Psi)) - d)|$. The *complementarity condition* that nulls the PD gap is therefore $|W_i(h(\Phi(\Psi))_i - d_i)| - x_i W_i(h(\Phi(\Psi)) - d) = 0 \quad \forall i$. the LS based PD framework can be written as

$$\sum_{i=1}^{M} |W_i(h(\Phi(\Psi)) - d_i)| - x^T W(h(\Phi(\Psi)) - d) = 0 \quad (12)$$

$$|x_i| \leq 1$$

$$J_{LS}^T(\Phi(\Psi))Wx + 2\alpha L^T L(\Phi(\Psi) - \Phi(\Psi)^{ast}) = 0$$

which constitutes the level set based PD method applied to the primal problem defined in (7). The smoothed version of LS based PD framework can be obtained through applying the *centering condition* which is the replacement of $|W_i(h(\Phi(\Psi)) - d_i)|$ by $\sqrt{(W_i(h(\Phi(\Psi)) - d_i)^2 + \beta}$, with $\beta > 0$. Replacing $|f_p(m)| = 0|$ and $\|g_d(m) = 0\|$ in (2-6), the smoothed LS based PD framework is achieved as

$$|x_i| \leq 1, \quad (13)$$

$$F_c(\Phi(\Psi)) = J_{LS}^T(\Phi(\Psi))Wx +$$

$$2\alpha M^T L^T L(\Phi(\Psi) - \Phi(\Psi^0)) = 0, \quad (14)$$

$$\mathrm{C}_d(\Phi(\Psi)) = (h_i(\Phi(\Psi)) - d_i) -$$

$$\mathrm{x}_i\sqrt{(h_i(\Phi(\Psi)) - d_i)^2 + \beta} = 0, \quad \beta > 0, \quad (15)$$

and the Gauss Newton method is applied to solve for the primal variables (Ψ) and the dual variables (x). To find the optimal solution of the above newton system, the derivatives of (15) and (14) with respect to $\partial\Psi$ and ∂x is calculated and the first order conditions are imposed. For (14), we have

$$\frac{\partial}{\partial\Psi}\left[J_{LS}^T(\Phi(\Psi))Wx + 2\alpha M^T L^T L(\Phi(\Psi) - \Phi(\Psi^0))\right] =$$

$$\frac{\partial}{\partial\Phi(\Psi)}\left[J_{LS}^T(\Phi(\Psi))Wx\right]\frac{\partial}{\partial\Psi}\left[\Phi(\Psi)\right] +$$

$$\frac{\partial}{\partial\Phi(\Psi)}\left[2\alpha M^T L^T L(\Phi(\Psi) - \Phi(\Psi^0))\right]\frac{\partial}{\partial\Psi}\left[\Phi(\Psi)\right] =$$

$$2\alpha M^T L^T LM, \quad (16)$$

and

$$\frac{\partial}{\partial x}\left[J_{LS}^T(\Phi(\Psi))Wx + 2\alpha M^T L^T L(\Phi(\Psi) - \Phi(\Psi^0))\right] =$$

$$\mathrm{J}_{LS}^T(\Phi(\Psi))W, \quad (17)$$

For (15), we have

$$\frac{\partial}{\partial\Psi}\left[(h_i(\Phi(\Psi)) - d_i) - x_i\sqrt{(h_i(\Phi(\Psi)) - d_i)^2 + \beta}\right] =$$

$$\mathrm{J}_{LS}(\Phi(\Psi)) - XE^{-1}FJ_{LS}(\Phi(\Psi)) = (I - XE^{-1}F)J_{LS}(\Phi(\Psi)) \quad (18)$$

where we define $f_i = h_i(\Phi(\Psi)) - d_i$, $F = diag(f_i)$, $X = diag(x_i)$, $\eta_i = \sqrt{f_i^2 + \beta}$, $E = diag(\eta_i)$, and I is the identity matrix. The partial derivatives of (15) with respect to ∂x are

$$\frac{\partial}{\partial x}\left[(h_i(\Phi(\Psi)) - d_i) - x_i\sqrt{(h_i(\Phi(\Psi)) - d_i)^2 + \beta}\right] = -E, \quad (19)$$

we assume the Jacobian ($J_{LS}(\Phi(\Psi))$ is constant and does not depend on the primal variables ($\Phi(\Psi)$) at every iteration of the LS based PD framework. The Newton system for solving the primal problem using the derived LS-PDIPM, which can also be obtained through applying the general solution in (6), is as follows

$$\begin{bmatrix} 2\alpha M^T L^T LM & J_{LS}^T W \\ (I - XE^{-1}F)J_{LS} & -E \end{bmatrix}\begin{bmatrix} \delta\Psi \\ \delta x \end{bmatrix} =$$

$$-\begin{bmatrix} J_{LS}^T Wx + 2\alpha M^T L^T L(\Phi(\Psi) - \Phi(\Psi^0)) \\ f - Ex \end{bmatrix} \quad (20)$$

The derived set of equations in (20) are iteratively solved for the primal variables ($\delta\Psi$) and the dual variables (δx) using an iterative method such as Newton method. A traditional line search procedure [6] can be applied to find an appropriate step length λ_Ψ resulting in the update $\Psi^{(k+1)} = \Psi^{(k)} + \lambda_\Psi\delta\Psi^{(k)}$, where k is the iteration number. A *scaling rule* is applied to compute the updates for the dual variables (x) [1].

4 Simulated Data

In our simulation, we apply the proposed LS-PDIPM to reconstruct images from the simulated data of EIT, as a highly ill-posed and challenging inverse problem. EIT image reconstruction is the process to produce a conductivity distribution image of a medium using the injection of electrical current into the medium and the collection of the resulting difference voltages across the electrodes attached to the boundary of the medium. We simulated EIT with 16 electrodes on one electrode plane discretized using a circular FEM. Figure 1 shows the used 2D phantom to generate simulated data with 1024 mesh elements. The phantom contains two sharp inclusions with the same conductivity located in the upper and the lower part of the mesh. The background conductivity value is 1 S/m and the inclusions have the conductivity of 0.9 S/m. The inverse problem used the mesh density of 576 elements, which was different than the mesh density of the forward problem (1024 elements).

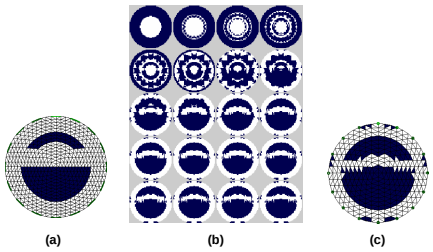

Figure 1: EIT Image reconstruction using the proposed LSRM. (a) 2D phantom. (b) The iterations of the LS-PDIPM (c) The final reconstructed image.

5 Results

We show the EIT reconstructed images of the proposed LS-PDIPM when applied to solve a primal problem with L1 norm on both the data mismatch and the regularization term (L1L1 problem). The reason of choosing the L1L1 formulation is because the L1 norms has been shown to offer the highest robustness to spatial noise and outliers [2]. The inverse solution is calculated using the proposed LS-PDIPM with $\beta = 1 \times 10^{-12}$. The stopping term to terminate the iterations depends on the value of the primal dual gap computed in every iteration of the LS-PDIPM. According to our experiments, 20 iterations for the LS-PDIPM were sufficient to reach to convergence. Figure 1(a) shows the applied 2D phantom with two inclusions with conductivity of 0.9 S/m. The reconstructed images for every iteration of the LS-PDIPM are shown in figure 1(b). The final reconstructed image using the LS-PDIPM is demonstrated in figure 1(c). Figure 2 compares the performance of the developed LS-PDIPM with four state of the art regularization methods, GN with Tikhonov regularization term, GN with NOSER algorithm, Total Variation (TV), and PDIPM with L1 norms on the data mismatch and regularization terms. To account for the possible systematic and random errors occurring in EIT data acquisition process, we consider four possible measurement conditions: 1) when there is no noise and data outliers (figure 2(a)), 2) with the presence of additive 60 dB Gaussian noise (figure 2(b)), 3) with the presence of strong data outliers (figure 2(c)), and 4) when there are both the 60 dB Gaussian noise and data outliers (figure 2(d)). GN methods (column 1 and 2 in figure 2) totally fails in the presence of noise and data outliers. TV is slightly robust to the additive noise; however, it fails in the presence of data outliers (column 3 in figure 2). PDIPM shows robust results when there exists either noise or data outliers; however, it fails when we perturb the EIT simulated data with 60 dB Gaussian noise plus a strong data outliers (column 4 in figure 2). The proposed LS-PDIPM is the winner method which is not highly suffered from the measurement noise and data outliers. The reconstruction images of the LS-PDIPM (column 4 in figure 2) are sharp images with little image artifacts when the EIT simulated data are perturbed by synthetic additive 60 dB Gaussian noise and strong data outliers.

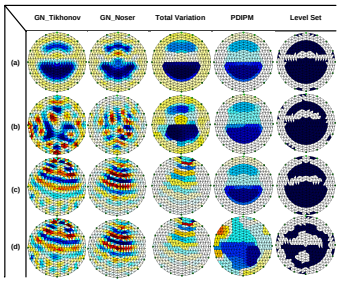

Figure 2: Comparison between the proposed LS-PDIPM and four state of the art regularization methods. (a) no additive noise and outliers. (b) with added 60 dB Gaussian noise. (c) with strong loss of data (outliers). (d) with 60 dB Gaussian noise and strong outliers.

6 Conclusion

We derive a novel level set based regularization method, called LS-PDIPM, that allows any possible combination of norms (L1 norm or L2 norm) on the data term and the regularization term. The proposed LS-PDIPM incorporates the benefit of applying level set technique to reconstruct sharp images as well as using the L1 norms in the formulation of inverse problems, which the later makes the reconstruction algorithm robust to noise and data outliers. We compared the LS-PDIPM with four state of the art regularization methods, GN with Tikhonov regularization term, GN with NOSER algorithm, Total Variation, and PDIPM. We show the LS-PDIPM performs better in reconstructing highly robust images against strong spatial noise and data outliers when compared with the competing methods (figure 2). We believe the LS-PDIPM can be successfully applied in a variety of inverse problem applications raised in engineering, geophysics, and life science. An extended version of the LS-PDIPM with several experimental results is currently under way.

References

[1] K. D. Andersen, E. Christiansen, A. Conn, and M. L. Overton. An efficient primal-dual interior-point method for minimizing a sum of Euclidean norms. *SIAM J. on Scientific Computing*, 22:243–262, 2000.

[2] A. Borsic and A. Adler. A primaldual interior-point framework for using the l1 or l2 norm on the data and regularization terms of inverse problems. *Journal of Inverse Problems*, 28, 2012.

[3] A. Borsic, B.M. Graham, A. Adler, and W.R.B. Lionheart. In Vivo Impedance Imaging with Total Variation Regularization. *IEEE Transaction on Medical Imaging*, 29:44–54, 2010.

[4] M. Cheney, D. Isaacson, J. C. Newell, S. Simake, and J. Goble. NOSER: An algorithm for solving the inverse conductivity problem. *Int J Imag Sys Technol*, 2:66–75, 1990.

[5] O Dorn and D Lesselier. Level set methods for inverse scattering-some recent developments. *Inverse Problems*, 25:125, 2009.

[6] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer, 1999.

[7] P. Rahmati, M. Soleimani, S. Pulletz, I. Frerichs, and A. Adler. Level-set-based reconstruction algorithm for eit lung images: first clinical results. *Journal on Physiological Measurement*, 33:739, 2012.

[8] A. N. Tikhonov. Solution of incorrectly formulated problems and the regularisation method. *Dokl. Akad. Nauk SSSR*, 151:501–504, 1963.